Computer representation of floating point numbers

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As the computer can only store finite numbers, an efficient way to store them is needed. Several problems may occur when dealing

with extreme small or large numbers on a computer.

If you don't have that understanding, get advice, take the time to learn, or [...] hope for the best.

— Bjarne Stroustrup.

Definition: A binary number is a finite sequence of digits $d_i \in \{0, 1\}$, i = 0, ..., n - 1. The value $\phi(d_i)$ of each digit is

defined as $\phi(d_i) = 2^i d_i$. The value $\phi(d)$ of a binary number $d = d_{n-1}d_{n-2} \dots d_1 d_0$ is given by

$$\phi(d) = \sum_{i=0}^{n-1} \phi(d_i)$$

smallest number: 0 largest number: $2^n - 1$

How can real numbers be represented? **Definition:** A *fixed-point* number consists of n + 1 are desired and k post desired digits

number consists of n + 1 pre-decimal and k post-decimal digits, $n, k \ge 0$.

A variety of systems exist to represent fixed-point numbers. Two examples are 'sign and magnitude' and 'two's complement'.

The highest bit is used as sign bit. The value $\phi(d)$ of a number $d = d_n d_{n-1} \dots d_0 \dots d_{-k}$ is given by

$$\phi(d) = (-1)^{d_n} \sum_{i=-k}^{n-1} \phi(d_i)$$

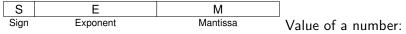
smallest number: $-(2^n - 2^{-k})$ largest number: $2^n - 2^{-k}$ **Note:** Two representations for 0 (e.g.

100 and 000) **Note:** Neighbouring numbers have distance 2^{-k} Some problems with fixed-point numbers:

- Very small and very large numbers cannot be represented
- Operations are not algebraically closed: 2^{n-1} is representable, $2^{n-1} + 2^{n-1}$ is not!
- Associative and distributive law are not applicable: $(2^{n-1}+2^{n-1})-2^{n-1}\neq 2^{n-1}+(2^{n-1}-2^{n-1})$

The position of the binary point is not fixed!

 \Longrightarrow Larger range of numbers with same number of digits.



$$(-1)^{S} \cdot M \cdot 2^{E}.$$

- Single precision (32 bit): S: 1 bit, E: 8 bit, M: 23 bit
- Double precision (64 bit): S: 1 bit, E: 11 bit, M: 52 bit

Observation: representation of a number is not unique. **Definition:** If $1 \le \phi(M) < 2$ the floating-point number is called

normalized, i.e. $M = 1.m_{-1}...m_{-k}$. The leading 1 needs not to be saved ("hidden bit"). For every normalized floating-point

number the value of *M* is calculated as $\phi(M) = 1 + \sum_{i=-1}^{-k} \phi(m_i)$. **Note:** 0 is not representable!

Note: Normalized numbers are unique.

IEEE 754 defines that

- the exponent bits are interpreted as an unsigned number $(E = e_{n-1} \dots e_0)$
- to be able to represent negative exponents, the so called *bias* is subtracted from E
- the bias B is 127 for single precision, 1023 for double precision $(B = 2^{n-1} 1)$

For *n* bits in the exponent the value of E is defined as $\phi(E) = \sum_{i=0}^{n-1} \phi(e_i) - B.$

Definition: If all exponent bits are 0, the hidden bit is also interpreted as 0. This way much smaller numbers can be represented. These numbers are called *denormalized*. The value of such a number is

$$\sum_{i=-1}^{-k} \phi(m_i) \cdot 2^{-126}$$

Note: 0 is representable again! If all bits in E are 1 and all bits in

M are 0, ∞ is represented.

	single precision	double precision
Sign bits	1	1
Exponent bits	8	11
Mantissa bits	23	52
Bits altogether	32	64
Bias	127	1023
Exponent range	-126 to 127	-1022 to 1023
Smallest normalized	2^{-126}	2^{-1022}
Largest normalized	$\sim 2^{128}$	$\sim 2^{1024}$
Smallest denormalized	2^{-149}	2^{-1074}
Decimal range	$\sim 10^{-38}$ to 10^{38}	$\sim 10^{-308}$ to 10^{308}

The distance from one number to the next varies from $\sim 10^{-45}$ to $\sim 10^{31}$ over the full range of single precision numbers. Definition:

This fact is expressed by the machine epsilon ε , which is the maximum relative error when representing a real number as a floating-point number. The machine epsilon is the smallest number such that $1 + \varepsilon \neq 1$ still holds. Only fractions whose denominator

is a power of 2 (e.g. $\frac{1}{2}$, $\frac{3}{8}$, $\frac{127}{256}$) can be represented exactly.

The properties of IEEE 754 numbers are

- numbers are unique when only normalized numbers are used
- not every number between the smallest and the largest is representable
- numbers are more dense around 0
- operations are still not algebraically closed.
- associative and distributive law are still not applicable!

The IEEE standard leaves it open, in which direction the numbers are stored in memory. Two possibilities are obvious:

- Big-endian, used by Sparc, Mac, PowerPC machines: The most significant byte is saved first
- Little-endian, used by Intel and Alpha machines: The least significant byte is saved first

The default length of some data types also depends on the architecture.

char:

signed: two's complement (8 bits) unsigned: binary number (8 bits) **int:**

signed: two's complement (16, 32 or 64 bits) unsigned: binary number (16, 32 or 64 bits) **long:**

signed: two's complement (32 or 64 bits) unsigned: binary number (32 or 64 bits)

Real numbers:

float: IEEE 754 with single precision (32 bit) double: IEEE 754 with double precision (64 bit) There also exists a

'long double' type, which provides IEEE 754 numbers with extended precision (80 bit).

These provide numbers up to an excess of $\sim 10^{4932}$ and are mainly needed for rounding-free calculations in the hardware itself, but can also be used within C++ programs.

Information about the used number representation can be obtained via the template numeric_limits<typename T> which is defined in the header <limits>:

- int radix base of exponent (usually 2 for binary)
- int digits number of bits in the mantissa
- T min() the minimal representable number
- T max() the maximal representable number
- T epsilon() the machine epsilon (ε)

- Operations are not algebraically closed.
- Associative and distributive law are not applicable.
- Not every real number is representable.
- Information about the used data types are available via numeric_limits.

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